Active Structure from Motion

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OUTLINE

REFERENCES

- Feature Depth Observation for Image-based Visual Servoing: Theory and Experiments, IJRR, 2008.
- A framework for active estimation: Application to structure from motion, CDC, 2013.
- Active Structure From Motion Application to Point, Sphere, and Cylinder, TRO, 2014.

Nonlinear Observation Scheme [1]

dynamical system in the form

$$\begin{cases} \dot{\boldsymbol{x}}_m = \boldsymbol{f}_m(\boldsymbol{x}_m, \boldsymbol{u}, t) + \boldsymbol{\Omega}^T(t)\boldsymbol{x}_u \\ \dot{\boldsymbol{x}}_u = \boldsymbol{f}_u(\boldsymbol{x}_m, \boldsymbol{x}_u, \boldsymbol{u}, t) \end{cases} \tag{1}$$

where $\boldsymbol{x}_m \in \mathbb{R}^m$ is the measurable component of the state, and $\boldsymbol{x}_u \in \mathbb{R}^p$ the unmeasurable component.
Consider the following observer

$$\begin{cases} \dot{\hat{\boldsymbol{x}}}_m = \boldsymbol{f}_m(\boldsymbol{x}_m, \boldsymbol{u}, t) + \boldsymbol{\Omega}^T(t)\hat{\boldsymbol{x}}_u + \boldsymbol{H}\xi \\ \dot{\hat{\boldsymbol{x}}}_u = \boldsymbol{f}_u(\boldsymbol{x}_m, \hat{\boldsymbol{x}}_u, \boldsymbol{u}, t) + \boldsymbol{\Lambda}\boldsymbol{\Omega}(t)\boldsymbol{P}\xi \end{cases} \tag{2}$$

$\xi = \boldsymbol{x}_m - \hat{\boldsymbol{x}}_m$, $z = \boldsymbol{x}_u - \hat{\boldsymbol{x}}_u$, $\boldsymbol{e} = [\xi^T, z^T]^T$

[1] Feature Depth Observation for Image-based Visual Servoing: Theory and Experiments, IJRR, 2008.

Nonlinear Observation Scheme

error dynamics

$$\begin{cases} \dot{\xi} = -\boldsymbol{H}\xi + \boldsymbol{\Omega}^T(t)\boldsymbol{z} \\ \dot{\boldsymbol{z}} = -\boldsymbol{\Lambda}\boldsymbol{\Omega}(t)\boldsymbol{P}\xi + (\boldsymbol{f}_u(\boldsymbol{x}_m, \boldsymbol{x}_u, \boldsymbol{u}, t) - \boldsymbol{f}_u(\boldsymbol{x}_m, \hat{\boldsymbol{x}}_u, \boldsymbol{u}, t)) \\ \quad = -\boldsymbol{\Lambda}\boldsymbol{\Omega}(t)\boldsymbol{P}\xi + \boldsymbol{g}(\boldsymbol{e}, t) \end{cases} \tag{3}$$

with $\boldsymbol{g}(\boldsymbol{e}, t)$ being a “perturbation term” vanishing w.r.t. the error vector $\boldsymbol{e}$, i.e., such that $\boldsymbol{g}(0, t) = 0, \forall t$.

Nonlinear Observation Scheme

Lemma 1 (Persistency of Excitation): Consider the system

$$\begin{cases} \dot{\xi} = -\boldsymbol{H}\xi + \boldsymbol{\Omega}^T(t)z \\ \dot{z} = -\boldsymbol{\Lambda}\boldsymbol{\Omega}(t)\boldsymbol{P}\xi \end{cases} \tag{4}$$

where $\boldsymbol{H} > 0$, $\boldsymbol{P} = \boldsymbol{P}^T > 0$ and $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^T > 0$. If $\|\boldsymbol{\Omega}\|(t)$ and $\|\dot{\boldsymbol{\Omega}}\|(t)$ are uniformly bounded and the *persistency of excitation* condition is satisfied, that is, there exists a $T > 0$ and $\gamma > 0$ such that

$$\int_t^{t+T} \boldsymbol{\Omega}(\tau)\boldsymbol{\Omega}^T(\tau)d\tau \geq \gamma \boldsymbol{I}_p > 0, \forall t \geq t_0 \tag{5}$$

then $(\xi, z) = (\mathbf{0}, \mathbf{0})$ is a globally exponentially stable equilibrium point.

Nonlinear Observation Scheme

The origin of (3) can be made globally exponentially stable[2]. If there exists a positive M such that $\|g(\boldsymbol{e},t)\| \leq M\|\boldsymbol{e}\|^2$,

$$\dot{V}(\boldsymbol{e},t) \leq -c_3\|\boldsymbol{e}\|^2 + \left\|\frac{\partial V}{\partial \boldsymbol{e}}\right\| \|g(\boldsymbol{e},t)\| \leq -c_3\|\boldsymbol{e}\|^2 + c_4 M\|\boldsymbol{e}\|^2 \quad (6)$$

If $m \geq p$, it is possible to instantaneously satisfy (5) by enforcing

$$\boldsymbol{\Omega}(t)\boldsymbol{\Omega}^T(t) \geq \frac{\gamma}{T}\boldsymbol{I}, \forall t. \quad (7)$$

[2] Feature Depth Observation for Image-based Visual Servoing: Theory and Experiments, IJRR, 2008.

Active Estimation Strategy [3]

$$\begin{cases} \dot{\xi} = -\boldsymbol{H}\xi + \boldsymbol{\Omega}^T(t)\boldsymbol{z} \\ \dot{\boldsymbol{z}} = -\boldsymbol{\Lambda}\boldsymbol{\Omega}(t)\boldsymbol{P}\xi + \boldsymbol{g}(\boldsymbol{e}, t) \end{cases} \tag{8}$$

Consider the following change of coordinates

$$\begin{cases} \tilde{\xi} = \boldsymbol{P}^{\frac{1}{2}}\xi \\ \tilde{\boldsymbol{z}} = \boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{z} \end{cases} \tag{9}$$

the system (8) takes the form

$$\begin{pmatrix} \dot{\tilde{\xi}} \\ \dot{\tilde{\boldsymbol{z}}} \end{pmatrix} = \left[\begin{pmatrix} \boldsymbol{0} & \tilde{\boldsymbol{\Omega}}^T(t) \\ -\tilde{\boldsymbol{\Omega}}(t) & \boldsymbol{0} \end{pmatrix} - \begin{pmatrix} \tilde{\boldsymbol{H}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \right] \begin{pmatrix} \tilde{\xi} \\ \tilde{\boldsymbol{z}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{g}} \end{pmatrix} \tag{10}$$

with $\tilde{\boldsymbol{H}} = \boldsymbol{P}^{\frac{1}{2}}\boldsymbol{H}\boldsymbol{P}^{-\frac{1}{2}}$, $\tilde{\boldsymbol{\Omega}}(t) = \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{\Omega}(t)\boldsymbol{P}^{\frac{1}{2}}$ and $\tilde{\boldsymbol{g}} = \boldsymbol{\Lambda}^{\frac{1}{2}}\boldsymbol{g}$.

[3] A framework for active estimation: Application to structure from motion, CDC, 2013.

Active Estimation Strategy

Neglect the presence of $\tilde{g}$ and analyze the dynamics of $\ddot{\tilde{z}}$

$$\dot{\tilde{z}} = -\tilde{\boldsymbol{\Omega}}(t)\tilde{\xi} \tag{11}$$

$$\begin{aligned}\ddot{\tilde{z}} &= -\dot{\tilde{\boldsymbol{\Omega}}}\tilde{\xi} - \tilde{\boldsymbol{\Omega}}\dot{\tilde{\xi}} \\ &= -\dot{\tilde{\boldsymbol{\Omega}}}\tilde{\xi} - \tilde{\boldsymbol{\Omega}}(-\tilde{\boldsymbol{H}}\tilde{\xi} + \tilde{\boldsymbol{\Omega}}^T\tilde{z}) \\ &= (\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{H}} - \dot{\tilde{\boldsymbol{\Omega}}})\tilde{\xi} - \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}^T\tilde{z} \\ &= (\dot{\tilde{\boldsymbol{\Omega}}}\tilde{\boldsymbol{\Omega}}^{\dagger} - \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{\Omega}}^{\dagger})\dot{\tilde{z}} - \tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}^T\tilde{z}\end{aligned} \tag{12}$$

with $\tilde{\boldsymbol{\Omega}}^{\dagger} \in \mathbb{R}^{m\times p}$ denoting the pseudo-inverse of $\tilde{\boldsymbol{\Omega}}$.

Active Estimation Strategy

$\tilde{\boldsymbol{\Omega}} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{V}}^T$, where $\tilde{\boldsymbol{\Sigma}} = [\tilde{\boldsymbol{S}}, \boldsymbol{0}]$, $\tilde{\boldsymbol{S}} = \mathrm{diag}(\tilde{\sigma}_i) \in \mathbb{R}^{p\times p}$ and $0 \leq \tilde{\sigma}_1 \leq \cdots \leq \tilde{\sigma}_p$.

As for $\dot{\tilde{\boldsymbol{\Omega}}}$ it is $\dot{\tilde{\boldsymbol{\Omega}}} = \dot{\tilde{\boldsymbol{U}}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{V}}^T + \tilde{\boldsymbol{U}}\dot{\tilde{\boldsymbol{\Sigma}}}\tilde{\boldsymbol{V}}^T + \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Sigma}}\dot{\tilde{\boldsymbol{V}}}^T$.

Denoting the skew-symmetric matrix $\tilde{\boldsymbol{\Gamma}}_U = \tilde{\boldsymbol{U}}^T\dot{\tilde{\boldsymbol{U}}}$ and $\tilde{\boldsymbol{\Gamma}}_V = \dot{\tilde{\boldsymbol{V}}}^T\tilde{\boldsymbol{V}}$.

$$\dot{\tilde{\boldsymbol{\Omega}}} = \tilde{\boldsymbol{U}}(\tilde{\boldsymbol{\Gamma}}_U\tilde{\boldsymbol{\Sigma}} + \dot{\tilde{\boldsymbol{\Sigma}}} + \tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\Gamma}}_V)\tilde{\boldsymbol{V}}^T \tag{13}$$

$$\begin{aligned}\dot{\tilde{\boldsymbol{\Omega}}}\tilde{\boldsymbol{\Omega}}^\dagger &= \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Gamma}}_U\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\Sigma}}^\dagger\tilde{\boldsymbol{U}}^T + \tilde{\boldsymbol{U}}\dot{\tilde{\boldsymbol{\Sigma}}}\tilde{\boldsymbol{\Sigma}}^\dagger\tilde{\boldsymbol{U}}^T + \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\boldsymbol{\Gamma}}_V\tilde{\boldsymbol{\Sigma}}^\dagger\tilde{\boldsymbol{U}}^T \\ &= \tilde{\boldsymbol{U}}(\tilde{\boldsymbol{\Gamma}}_U + \dot{\tilde{\boldsymbol{S}}}\tilde{\boldsymbol{S}}^{-1} + \tilde{\boldsymbol{S}}\bar{\boldsymbol{\Gamma}}_V\tilde{\boldsymbol{S}}^{-1})\tilde{\boldsymbol{U}}^T\end{aligned} \tag{14}$$

$\bar{\boldsymbol{\Gamma}}_V = -\bar{\boldsymbol{\Gamma}}_V^T$ is the $p \times p$ upper-left block of matrix $\tilde{\boldsymbol{\Gamma}}_V$.

Active Estimation Strategy

The matrix $\tilde{\boldsymbol{H}}$ is designed as

$$\tilde{\boldsymbol{H}} = \tilde{\boldsymbol{V}} \begin{bmatrix} \boldsymbol{D}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D}_2 \end{bmatrix} \tilde{\boldsymbol{V}}^T \tag{15}$$

with $\boldsymbol{D}_1 \in \mathbb{R}^{p \times p} > 0$.
This choice yields

$$\tilde{\boldsymbol{\Omega}} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{\Omega}}^{\dagger} = \tilde{\boldsymbol{U}} \tilde{\boldsymbol{S}} \boldsymbol{D}_1 \tilde{\boldsymbol{S}}^{-1} \tilde{\boldsymbol{U}}^T \tag{16}$$

Active Estimation Strategy

finally

$$\begin{aligned}\ddot{\tilde{z}} &= \tilde{U}(\tilde{\Gamma}_U + \dot{\tilde{S}}\tilde{S}^{-1} + \tilde{S}\bar{\Gamma}_V\tilde{S}^{-1} - \tilde{S}D_1\tilde{S}^{-1})\tilde{U}^T\dot{\tilde{z}} - \tilde{U}\tilde{S}^2\tilde{U}^T\tilde{z} \\ &= (\tilde{U}\tilde{S})(\tilde{S}^{-1}\tilde{\Gamma}_U\tilde{S} + \dot{\tilde{S}}\tilde{S}^{-1} + \bar{\Gamma}_V - D_1)(\tilde{S}^{-1}\tilde{U}^T)\dot{\tilde{z}} - \tilde{U}\tilde{S}^2\tilde{U}^T\tilde{z} \\ &= (\tilde{U}\tilde{S})(\tilde{\Pi} - D_1)(\tilde{S}^{-1}\tilde{U}^T)\dot{\tilde{z}} - (\tilde{U}\tilde{S})\tilde{S}^2(\tilde{S}^{-1}\tilde{U}^T)\tilde{z}\end{aligned} \tag{17}$$

where

$$\tilde{\Pi} = \tilde{S}^{-1}\tilde{\Gamma}_U\tilde{S} + \dot{\tilde{S}}\tilde{S}^{-1} + \bar{\Gamma}_V \tag{18}$$

Active Estimation Strategy

Consider a change of coordinates

$$\eta = (\tilde{\boldsymbol{S}}^{-1}\tilde{\boldsymbol{U}}^T)\tilde{z} \tag{19}$$

in the approximation $\tilde{\boldsymbol{S}}^{-1}\tilde{\boldsymbol{U}}^T \approx const$, the system takes the simple form

$$\ddot{\eta} = (\tilde{\boldsymbol{\Pi}} - \boldsymbol{D}_1)\dot{\eta} - \tilde{\boldsymbol{S}}^2\eta \tag{20}$$

which is a (unit-)mass-spring-damper system with diagonal stiffness matrix $\tilde{\boldsymbol{S}}^2$.

Active Estimation Strategy

The convergence rate of (20) is related to the slowest mode of the system, i.e., that associated to the element $\tilde{\sigma}_1^2$ in $\tilde{\boldsymbol{S}}^2$.
To impose a desired transient response to $\eta(t)$, one can "place the holes" of (20) by

- regulating $\tilde{\sigma}_1^2$ to a desired value $\tilde{\sigma}_{1,des}^2$,
- shaping the damping factor $\boldsymbol{D}_1$ to prevent the occurrence of oscillatory modes,

Active Estimation Strategy

Shaping the damping factor $\boldsymbol{D}_1$

A reasonable choice for $\boldsymbol{D}_1$ could be $\boldsymbol{D}_1 = \tilde{\boldsymbol{\Pi}} + \boldsymbol{C}$, with $\boldsymbol{C}$ any positive definite matrix, such as a diagonal one $\boldsymbol{C} = \mathrm{diag}(c_i), c_i > 0$, so as to obtain a decoupled transient behavior

$$\ddot{\eta}_i + c_i \dot{\eta}_i + \tilde{\sigma}_i^2 \eta_i = 0, i = 1 \dots p \tag{21}$$

Taking $c_i = c_i^* = 2\tilde{\sigma}_i$ imposes a *critically damped* evolution to the estimation error.

However, for any arbitrary pair $(\boldsymbol{C}, \tilde{\boldsymbol{\Pi}})$, $\boldsymbol{D}_1$ may not necessarily remain positive definite over time.

Active Estimation Strategy

Shaping the damping factor $\boldsymbol{D}_1$

By suitably bounding $\|\tilde{\mathbf{\Pi}}\| \leq q\boldsymbol{I}$, any $\boldsymbol{C} > q\boldsymbol{I}$ could guarantee $\boldsymbol{D}_1 > 0$. However, this possibility results in an *over-damped transient response* for the system, since in the general case, $\boldsymbol{C} > q\boldsymbol{I} > \mathrm{diag}(c_i^*)$.

Therefore, the effects of $\tilde{\mathbf{\Pi}}$ on the transient by just taking $\boldsymbol{D}_1 = \mathrm{diag}(c_i^*) > 0$.

Active Estimation Strategy

Tuning the stiffness matrix $\tilde{S}^2$

$\tilde{S}^2 = \mathrm{diag}(\tilde{\sigma}_i^2)$ contains the p eigenvalues of $\tilde{\Omega}\tilde{\Omega}^T$ and $S^2 = \mathrm{diag}(\sigma_i^2)$ the eigenvalues of $\Omega\Omega^T$ in the original coordinates (ξ, z).

$$\tilde{\Omega}\tilde{\Omega}^T = \Lambda^{\frac{1}{2}}\Omega P \Omega^T \Lambda^{\frac{1}{2}} \tag{22}$$

The gains P, Λ can be exploited to amplify/attenuate the eigenvalues of $\tilde{S}^2$.

Also, one needs to ensure a minimum threshold $\sigma_1^2(t) \geq \sigma_{min}^2 > 0$ for the estimation to converge, i.e., for fulfilling the PE condition. This can be achieved by actively tuning matrix S^2.

Active Estimation Strategy

Tuning the stiffness matrix $\tilde{S}^2$

Assuming $\boldsymbol{P} = \alpha \boldsymbol{I}$ and $\boldsymbol{\Lambda} = \beta \boldsymbol{I}$, $\alpha > 0, \beta > 0$ yields $\tilde{\sigma}_i^2 = \alpha\beta\sigma_i^2$. Therefore, seeking a desired value $\tilde{\sigma}_i^2$ is equivalent to imposing

$$\sigma_i^2 \to \sigma_{i,des}^2 = \frac{\tilde{\sigma}_{i,des}^2}{\alpha\beta} \tag{23}$$

One can then focus on the regulation of σ_i^2.

Active Estimation Strategy

Tuning the stiffness matrix $\tilde{S}^2$

An explicit expression of the time derivative of $\sigma_i^2(t) = \sigma_i^2(\boldsymbol{x}_m, \boldsymbol{u}(t))$ can be obtained[4]

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_i^2(t) = \sum_{j=1}^{v}\left(\boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial u_j}\boldsymbol{v}_i \dot{u}_j\right) + \sum_{j=1}^{n}\left(\boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial x_{m_j}}\boldsymbol{v}_i \dot{x}_{m_j}\right) \tag{24}$$

where $\boldsymbol{v}_i \in \mathbb{R}^p$ is the normalized eigenvector associated to σ_i^2.

$$\boldsymbol{J}_{u,i} = \begin{bmatrix} \boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial u_1}\boldsymbol{v}_i & \cdots & \boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial u_v}\boldsymbol{v}_i \end{bmatrix} \in \mathbb{R}^{1\times v} \tag{25}$$

$$\boldsymbol{J}_{u,i} = \begin{bmatrix} \boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial x_{m_1}}\boldsymbol{v}_i & \cdots & \boldsymbol{v}_i^T \frac{\partial(\boldsymbol{\Omega}\boldsymbol{\Omega}^T)}{\partial x_{m_n}}\boldsymbol{v}_i \end{bmatrix} \in \mathbb{R}^{1\times n} \tag{26}$$

[4] Estimating the Jacobian of the Singular Value Decomposition: Theory and Application, ECCV, 2000.

Active Estimation Strategy

Tuning the stiffness matrix $\tilde{S}^2$
Eq. (24) can be rewritten as

$$\dot{(\sigma_i^2)} = \boldsymbol{J}_{u,i}\dot{\boldsymbol{u}} + \boldsymbol{J}_{x,i}\dot{\boldsymbol{x}}_m \tag{27}$$

Any *differential inversion technique* can be applied to (27) in order to affect the behavior of the i-th eigenvalue σ_i^2 by acting upon vector $\dot{\boldsymbol{u}}$.
It is in general not possible to fully compensate for the term $\boldsymbol{J}_{x,i}\dot{\boldsymbol{x}}_m$ because of a direct dependence of $\dot{\boldsymbol{x}}_m$ from the unmeasurable $\boldsymbol{x}_u$.

Depth Estimation for a Point Feature [5]

Let $\boldsymbol{p} = (x, y, 1) = (X/Z, Y/Z, 1) \in \mathbb{R}^3$ be the perspective projection of a 3D point $\boldsymbol{P} = (X, Y, Z)$ onto the image plane of a pinhole camera. The differential relationship between the image motion of a point feature and the camera linear/angular velocity $\boldsymbol{u} = (\boldsymbol{v}, \boldsymbol{\omega}) \in \mathbb{R}^6$ expressed in camera frame is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{1}{Z} & 0 & \frac{x}{Z} & xy & -(1+x^2) & y \\ 0 & -\frac{1}{Z} & \frac{y}{Z} & 1+y^2 & -xy & -x \end{bmatrix} \boldsymbol{u} \tag{28}$$

where Z is the depth of the feature point. The dynamics of Z is

$$\dot{Z} = \begin{bmatrix} 0 & 0 & -1 & -yZ & xZ & 0 \end{bmatrix} \boldsymbol{u} \tag{29}$$

[5] Active Structure From Motion Application to Point, Sphere, and Cylinder, TRO, 2014.

Depth Estimation for a Point Feature

By defining $\boldsymbol{x}_m = (x, y)$ and $x_u = 1/Z$ with $m = 2$ and $p = 1$, we obtain for (1)

$$\begin{cases} \boldsymbol{f}_m(\boldsymbol{x}_m, \boldsymbol{u}, t) = \begin{bmatrix} xy & -(1+x^2) & y \\ 1+y^2 & -xy & -x \end{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{\Omega}(\boldsymbol{x}_m, \boldsymbol{v}) = \begin{bmatrix} xv_z - v_x & yv_z - v_y \end{bmatrix} \\ f_u(\boldsymbol{x}_m, x_u, \boldsymbol{u}, t) = v_x x_u^2 + (y\omega_x - x\omega_y) x_u \end{cases} \tag{30}$$

with the perturbation term $g(\boldsymbol{e}, t)$ in (3)

$$g(\boldsymbol{e}, t) = v_z(x_u^2 - \hat{x}_u^2) + (y\omega_x - x\omega_y) z \tag{31}$$

Depth Estimation for a Point Feature

In the point feature case, matrix $\boldsymbol{\Omega}\boldsymbol{\Omega}^T$ reduces to its single eigenvalue

$$\sigma_1^2 = \|\boldsymbol{\Omega}\|^2 = (xv_z - v_x)^2 + (yv_z - v_y)^2 \tag{32}$$

then the Jacobian $\boldsymbol{J}_{u,1}$ in (27) is given by

$$\boldsymbol{J}_{u,1} = 2 \begin{bmatrix} xv_z - v_x \\ yv_z - v_y \\ (xv_z - v_x)x + (yv_z - v_y)y \\ 0 \\ 0 \\ 0 \end{bmatrix}^T = \begin{bmatrix} \boldsymbol{J}_{v,1} & \mathbf{0} \end{bmatrix} \tag{33}$$

Depth Estimation for a Point Feature

Some remarks:

- σ_1^2 does not depend on ω, it is possible to *freely exploit* the camera angular velocity for fulfilling additional goals of interest. For example, one can use ω for keeping $\boldsymbol{x}_m \simeq const$ in order to render the effect of $\dot{\boldsymbol{x}}_m$ in (27).
- $\boldsymbol{J}_{v,1}\boldsymbol{p} = 0$: the derivative of σ_1^2 is orthogonal to projection ray passing through $\boldsymbol{p}$.

Depth Estimation for a Point Feature

The value of σ_1^2 directly affects the convergence speed of the estimation error. What conditions on $\boldsymbol{p}$ and $\boldsymbol{v}$ result in the largest possible σ_1^2?
Let $\boldsymbol{e}_3 = (0, 0, 1)$ be the camera optical axis,

$$\begin{bmatrix} \boldsymbol{\Omega}^T \\ 0 \end{bmatrix} = [\boldsymbol{e}_3]_\times [\boldsymbol{p}]_\times \boldsymbol{v} \tag{34}$$

Therefore

$$\begin{aligned} \sigma_1^2 &= \begin{bmatrix} \boldsymbol{\Omega}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}^T \\ 0 \end{bmatrix} = \| [\boldsymbol{e}_3]_\times [\boldsymbol{p}]_\times \boldsymbol{v} \|^2 \\ &= \|\boldsymbol{p}\|^2 \|\boldsymbol{v}\|^2 \sin^2(\theta_{p,v}) \sin^2\left(\theta_{e_3,[p]_\times v}\right) \end{aligned} \tag{35}$$

Depth Estimation for a Point Feature

The maximum attainable value for σ_1^2 is

$$\sigma_{\max}^2 = \max_{p,v} \sigma_1^2 = \|\boldsymbol{p}\|^2 \|\boldsymbol{v}\|^2 \tag{36}$$

The maximum is obtained when

$$\begin{bmatrix} \boldsymbol{p}^T \\ \boldsymbol{e}_3^T [\boldsymbol{p}]_\times \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} x & y & 1 \\ -y & x & 0 \end{bmatrix} \boldsymbol{v} = 0 \tag{37}$$

If $\boldsymbol{p} \neq \boldsymbol{e}_3$ (point feature *not* at the center the image plane), system (37) has (full) rank 2 and admits the unique solution (up to a scalar factor)

$$\boldsymbol{v} = \delta [\boldsymbol{p}]_\times^2 \boldsymbol{e}_3, \quad \delta \in \mathbb{R} \tag{38}$$

Depth Estimation for a Point Feature

$$\boldsymbol{v} = \delta[\boldsymbol{p}]_\times^2 \boldsymbol{e}_3, \quad \delta \in \mathbb{R}$$

This requires $\boldsymbol{v}$ to be orthogonal to $\boldsymbol{p}$ and to lie on the plane defined by vectors $\boldsymbol{p}$ and $\boldsymbol{e}_3$.
If $\boldsymbol{p} = \boldsymbol{e}_3$ (point feature at the center of the image plane), system (37) loses rank and any $\boldsymbol{v} \perp \boldsymbol{e}_3$ is a valid solution.

Depth Estimation for a Point Feature

Conclusion:
For a given norm of the linear velocity $\|\boldsymbol{v}\|$, system (37) determines the direction of $\boldsymbol{v}$ resulting in $\sigma_1^2 = \sigma_{\max}^2$.
The value of $\sigma_{\max}^2$ is also a function of the feature point location $\boldsymbol{p}$ that can be arbitrarily positioned on the image plane.
$\sigma_{\max}^2 = \|\boldsymbol{v}\|^2$ for $\boldsymbol{p} = \boldsymbol{e}_3$ and $\sigma_{\max}^2 = \|\boldsymbol{p}\|^2\|\boldsymbol{v}\|^2 > \|\boldsymbol{v}\|^2\ \forall \boldsymbol{p} \neq \boldsymbol{e}_3$.

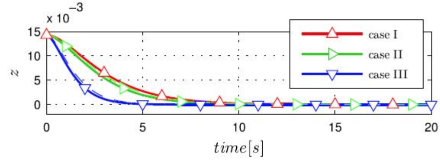
Depth Estimation for a Point Feature

- The smallest $\sigma^2_{\max}$ (the slowest "optimal" convergence for the depth estimation error) is obtained for the smallest value of $\|\boldsymbol{p}\|$, i.e., $\boldsymbol{p} = \boldsymbol{e}_3$ (feature point at the center of the image plane). In this case $v_z = 0$ ($\boldsymbol{v} \perp \boldsymbol{p}$), the camera moves on the surface of a sphere with a constant radius (depth) pointing at the feature point. And $g(\boldsymbol{e}, t) \equiv 0$ and global convergence is achieved.
- The largest $\sigma^2_{\max}$ (the fastest "optimal" convergence) is obtained for the largest possible value of $\|\boldsymbol{p}\|$. However, this results in $g(\boldsymbol{e}, t) \neq 0$ and only local convergence is achieved.

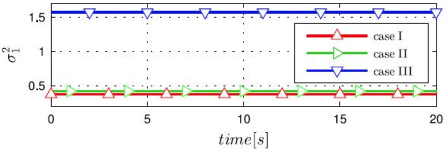
Simulation

- A constant $v(t) \equiv v(t_0) = const$ is kept during motion with $v(t_0)$ being a solution of (37).
- Consider three cases,
  - case I: the point feature is kept at the center of the image plane (red line),
  - case II: the point feature is kept at one of the corners of an image plane with the same size of the camera used in the experiments (green line).
  - case III: the point feature is kept at one of the corners of an image plane with a size five times larger than case II (blue line).

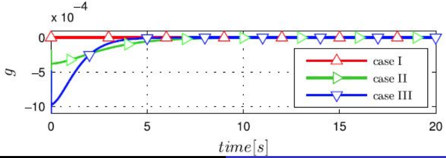
Simulation



(a)



(b)



Experiment

- $\|\boldsymbol{v}\|$ is kept constant over time.
- The angular velocity input ω is exploited to keep $\boldsymbol{s} \simeq (0,0)$ over time (point feature kept at the center the image plane).
- Control law

$$\dot{\boldsymbol{v}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|^2} k_1 (\kappa_{des} - \kappa) + k_2 \left(\boldsymbol{I}_3 - \frac{\boldsymbol{v}\boldsymbol{v}^T}{\|\boldsymbol{v}\|^2} \right) \boldsymbol{J}_{v,1}^T \tag{39}$$

  with $k_1 > 0$, $k_2 \geq 0$, $\kappa = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{v}$, $\kappa_{des} = \frac{1}{2}\boldsymbol{v}_0^T\boldsymbol{v}_0$.
- Consider two cases,
  - case I: $\sigma_1^2(t)$ is actively maximized ($k_2 = 0$, red line),
  - case II: constant velocity $\boldsymbol{v}(t) = \boldsymbol{v}_0 = const$ ($k_2 > 0$, blue line).

Experiment

- case I: $\sigma_1^2(t)$ is actively maximized ($k_2 = 0$, red line),
- case II: constant velocity $v(t) = v_0 = const$ ($k_2 > 0$,

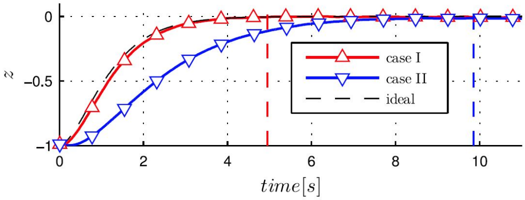


Figure: Behavior of the estimation error.

Experiment

- case I: $\sigma_1^2(t)$ is actively maximized ($k_2 = 0$, red line),
- case II: constant velocity $\boldsymbol{v}(t) = \boldsymbol{v}_0 = const$ ($k_2 > 0$,

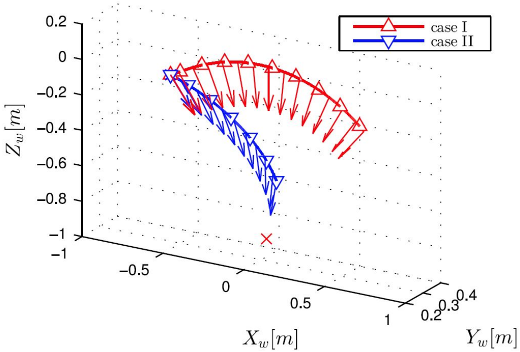


Figure: Camera trajectories.

Experiment

- case I: $\sigma_1^2(t)$ is actively maximized ($k_2 = 0$, red line),
- case II: constant velocity $\boldsymbol{v}(t) = \boldsymbol{v}_0 = const$ ($k_2 > 0$,

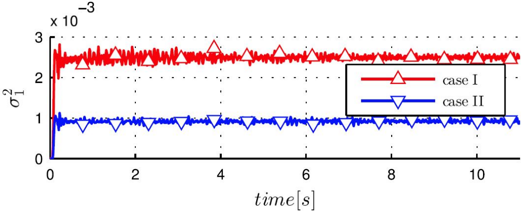


Figure: Behavior of $\sigma_1^2(t)$.