Plane Fusion

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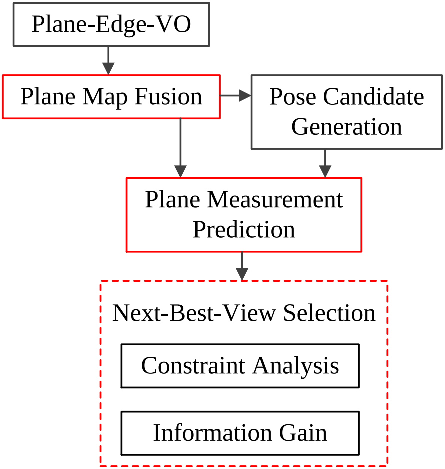
System Overview



OUTLINE

Homogeneous Representation of Planes

- The plane equation $\boldsymbol{n}^T\boldsymbol{p}+d=0$ is unaffected by multiplication by a non-zero scalar.
- The plane can be represented as a homogeneous vector $\Pi=[\Pi_1,\Pi_2,\Pi_3,\Pi_4]^T\in\mathbb{P}^3$ in projective space. [1]

$$\Pi^T\boldsymbol{P}=0 \tag{1}$$

$\boldsymbol{P}=[p_1,p_2,p_3,p_4]\in\mathbb{P}^3$ is the homogeneous coordinates corresponding to the point $\boldsymbol{p}$.

$$\begin{aligned}\boldsymbol{n}&=\frac{[\Pi_1,\Pi_2,\Pi_3]^T}{\sqrt{\Pi_1^2+\Pi_2^2+\Pi_3^2}}\\ d&=\frac{\Pi_4}{\sqrt{\Pi_1^2+\Pi_2^2+\Pi_3^2}}\end{aligned} \tag{2}$$

[1] R. Hartley and A. Zisserman, Multiple View Geometry in Computer Vision. Cambridge University Press, 2003, second Edition.

Homogeneous Representation of Planes

- Spherically normalizing the homogeneous vector $\Pi = [\Pi_1, \Pi_2, \Pi_3, \Pi_4]^T \in \mathbb{P}^3$ yields $\pi = \Pi / \|\Pi\| \in \mathbb{S}^3$.
- $\mathbb{S}^3$ is the 3-sphere in the space $\mathbb{R}^4$, which is a Lie group under the operation of quaternion multiplication when its elements are viewed as unit quaternions.
- The group $\mathbb{S}^3$ is also known as the *special unitary group* $\mathbb{SU}(2)$.
- For any $\pi = [\pi_1, \pi_2, \pi_3, \pi_4]^T \in \mathbb{S}^3$, it can be written as [2]

$$\boldsymbol{Q} = \begin{bmatrix} \pi_1 + \pi_4 i & \pi_2 + \pi_3 i \\ -\pi_2 + \pi_3 i & \pi_1 - \pi_4 i \end{bmatrix} \in \mathbb{SU}(2), \det \boldsymbol{Q} = 1 \tag{3}$$

[2] J. Stillwell, Naive Lie Theory. Undergraduate Texts in Mathematics, 2008.

Uncertainty Representation for Homogeneous Entities

Problems of homogeneous entities in fusion and optimization: [3]

- The redundant representation of homogeneous entities (planes) results in the *scale ambiguity* (often avoided by proper normalization).
- The covariance matrices of the spherically normalized homogeneous vectors are singular due to the homogeneity.
- The redundant representation also requires additional constraints in optimization.

[3] 2010, ACCV, Minimal representations for uncertainty and estimation in projective spaces.

Minimal Representation of Uncertainty for Planes

- Define random variables for $\mathbb{SU}(2)$

$$\pi = \exp(\zeta^{\wedge})\bar{\pi} \tag{4}$$

where $\bar{\pi}$ is a noise-free value and $\zeta \in \mathbb{R}^3$ is a noisy perturbation.

- $\wedge$ operator: turns $\zeta \in \mathbb{R}^3$ into a member of $\mathfrak{su}(2)$.

$$\zeta^{\wedge} = \zeta_1 \mathbf{i} + \zeta_2 \mathbf{j} + \zeta_3 \mathbf{k} \tag{5}$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a basis of $\mathfrak{su}(2)$ [4]

$$\mathbf{i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}. \tag{6}$$

The $\vee$ operator is the inverse of $\wedge$.

[4] J. Stillwell, Naive Lie Theory. Undergraduate Texts in Mathematics, 2008.

Minimal Representation of Uncertainty for Planes

- Directly define the probability density function (PDF) in the vectorspace $\mathbb{R}^3$. [5]

$$p(\zeta) = \mathcal{N}(\mathbf{0}, \Sigma) \tag{7}$$

  where Σ is a 6×6 covariance matrix.
- It induces a PDF over $\mathbb{SU}(2)$, $p(\pi)$.

[5] 2014, TRO, Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problems.

Minimal Representation of Uncertainty for Planes

- Derivation of $p(\pi)$. [6]

$$\int_{\mathbb{R}^3} p(\zeta)d\zeta = \int_{\mathbb{R}^3} \eta \exp\left(-\frac{1}{2}\zeta^T\Sigma^{-1}\zeta\right)d\zeta = 1 \tag{8}$$

where $\eta = \frac{1}{\sqrt{(2\pi)^3 \det\Sigma}}$.

$$\delta\pi = \ln\left(\pi'\pi^{-1}\right)^{\vee} \tag{9}$$

where

$$\pi' = \exp\left((\zeta+\delta\zeta)^{\wedge}\right)\bar{\pi}, \pi = \exp\left(\zeta^{\wedge}\right)\bar{\pi} \tag{10}$$

$$\delta\pi = \boldsymbol{J}\delta\zeta \tag{11}$$

where $\boldsymbol{J}$ is the Jacobian for $\mathbb{SU}(2)$ (will be calculated later).

[6] 2014, TRO, Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problems.

Minimal Representation of Uncertainty for Planes

- An infinitesimal volume of $\mathbb{SU}(2)$ $d\pi$ and the perturbation in ζ can be related by

$$d\pi = |\det(\boldsymbol{J})|\, d\zeta \qquad (12)$$

- The PDF over π is found by

$$\begin{aligned} 1 &= \int_{\mathbb{R}^3} \eta \exp\left(-\frac{1}{2}\zeta^T \Sigma^{-1} \zeta\right) d\zeta \\ &= \int_{\mathbb{SU}(2)} \beta \exp\left(-\frac{1}{2} \ln(\pi\bar{\pi}^{-1})^{\vee T} \Sigma^{-1} \ln(\pi\bar{\pi}^{-1})^{\vee}\right) d\pi \qquad (13) \\ &= \int_{\mathbb{SU}(2)} p(\pi) d\pi \end{aligned}$$

  where

$$\beta = \frac{\eta}{|\det(\boldsymbol{J})|} \qquad (14)$$

- $p(\pi)$ is not Gaussian because β depends on π via $\boldsymbol{J}$.

Calculation of Jacobian for $\mathbb{SU}(2)$

- Baker-Campell-Hausdorff (BCH) Formula

$$\begin{aligned}\ln\left(\exp(\boldsymbol{A})\exp(\boldsymbol{B})\right) =& \boldsymbol{A}+\boldsymbol{B}+\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]+\frac{1}{12}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] \\ &+\frac{1}{12}[\boldsymbol{B},[\boldsymbol{B},\boldsymbol{A}]]-\frac{1}{24}[\boldsymbol{B},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]]+\cdots\end{aligned} \tag{15}$$

where the Lie bracket is given by

$$[\boldsymbol{A},\boldsymbol{B}]=\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}. \tag{16}$$

- The BCH formula is an infinite series. If only the terms linear in $\boldsymbol{A}$ are kept,

$$\ln\left(\exp(\boldsymbol{A})\exp(\boldsymbol{B})\right)\approx\boldsymbol{B}+\sum_{n=0}^{\infty}\frac{B_n}{n!}[\boldsymbol{B},[\boldsymbol{B},\cdots[\boldsymbol{B},\boldsymbol{A}]\cdots]] \tag{17}$$

where B_n are Bernoulli numbers.

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30},\cdots \tag{18}$$

Calculation of Jacobian for $\mathbb{SU}(2)$

- In the case of $\mathbb{SU}(2)$, $\zeta_a = [\zeta_{a1}, \zeta_{a2}, \zeta_{a3}]^T$, $\zeta_b = [\zeta_{b1}, \zeta_{b2}, \zeta_{b3}]^T$

$$
\begin{aligned}
[\zeta_a^\wedge, \zeta_b^\wedge] &= \zeta_a^\wedge \zeta_b^\wedge - \zeta_b^\wedge \zeta_a^\wedge \\
&= 2\mathbf{i}(\zeta_{a2}\zeta_{b3} - \zeta_{a3}\zeta_{b2}) + 2\mathbf{j}(\zeta_{a3}\zeta_{b1} - \zeta_{a1}\zeta_{b3}) + 2\mathbf{k}(\zeta_{a1}\zeta_{b2} - \zeta_{a2}\zeta_{b1}) \\
&= 2\zeta_a \times \zeta_b = ([2\zeta_a]_\times \zeta_b)^\wedge
\end{aligned} \tag{19}
$$

  Note that the cross product is defined on the basis of $\mathfrak{su}(2)$.

- Applying the BCH formula and assuming that ζ_a is small (perturbation),

$$
\ln(\pi_a \pi_b)^\vee = \ln(\exp(\boldsymbol{A})\exp(\boldsymbol{B}))^\vee \approx \zeta_b + \boldsymbol{J}_b^{-1}\zeta_a \tag{20}
$$

  where

$$
\boldsymbol{J}_b^{-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}([2\zeta_a]_\times)^n \tag{21}
$$

Calculation of Jacobian for $\mathbb{SU}(2)$

- then

$$\begin{aligned} \boldsymbol{J}_b &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left([2\zeta_a]_\times\right)^n \\ &= \frac{\sin\phi}{\phi}\boldsymbol{I} + \left(1 - \frac{\sin\phi}{\phi}\right)\boldsymbol{m}\boldsymbol{m}^T + \frac{1-\cos\phi}{\phi}[\boldsymbol{m}]_\times \end{aligned} \tag{22}$$

where

$$\begin{aligned} \phi &= \|2\zeta_b\| \\ \boldsymbol{m} &= \frac{2\zeta_b}{\phi} \end{aligned} \tag{23}$$

- The $\boldsymbol{J}_b$ is of the similar form to the Jacobian for $\mathbb{SO}(3)$ (the Eq.(98) in TRO2014 [7]).

[7] 2014, TRO, Associating Uncertainty With Three-Dimensional Poses for Use in Estimation Problems.

Plane Fusion

- Suppose that there are N estimates of a plane $\bar{\pi}_i, \Sigma_i, i = 1, 2, \cdots, N$, and fuse them into a single optimal estimate $\bar{\pi}^*, \Sigma^*$.
- The error between the optimal estimate $\bar{\pi}^*$ and the individual estimate is defined as $\boldsymbol{e}_i \sim \mathcal{N}(\mathbf{0}, \Sigma_i)$.

$$\begin{aligned} \boldsymbol{e}_i &= \ln\left(\bar{\pi}^*\bar{\pi}_i^{-1}\right)^{\vee} = \ln\left(\exp(\zeta^{\wedge})\bar{\pi}\bar{\pi}_i^{-1}\right)^{\vee} \\ &= \ln\left(\exp(\zeta^{\wedge})\exp(\zeta_k^{\wedge})\right)^{\vee} \end{aligned} \tag{24}$$

where $\bar{\pi}$ is the current guess.

- Applying (20),

$$\boldsymbol{e}_i \approx \zeta_i + \boldsymbol{J}_i^{-1}\zeta. \tag{25}$$

where $\boldsymbol{J}_i$ is calculated in (22).

Plane Fusion

- The cost function is defined as

$$V = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^T \Sigma_i^{-1} \boldsymbol{e}_i \approx \frac{1}{2}\sum_{i=1}^{N} \left(\zeta_i + \boldsymbol{J}_i^{-1}\zeta\right)^T \Sigma_i^{-1} \left(\zeta_i + \boldsymbol{J}_i^{-1}\zeta\right) \tag{26}$$

- In each iteration ζ is solved by

$$\zeta = -\left(\sum_{i=1}^{N} \boldsymbol{J}_i^{-T}\Sigma_i^{-1}\boldsymbol{J}_i^{-1}\right)^{-1} \left(\sum_{i=1}^{N} \boldsymbol{J}_i^{-T}\Sigma_i^{-1}\zeta_i\right) \tag{27}$$

and the solution of the plane parameter is updated by

$$\bar{\pi} \leftarrow \exp \zeta^{\wedge} \bar{\pi} \tag{28}$$

- At the last iteration, the covariance of the estimate is calculated by

$$\Sigma = \left(\sum_{i=1}^{N} \boldsymbol{J}_i^{-T}\Sigma_i^{-1}\boldsymbol{J}_i^{-1}\right)^{-1} \tag{29}$$

Plane Measurement Prediction

- Define a "descriptor" for a plane landmark in the map.

$$D_\pi = \{\boldsymbol{p}_{\pi c}, \boldsymbol{p}_{\pi b1}, \boldsymbol{p}_{\pi b2}, \cdots, \boldsymbol{p}_{\pi b N_b}\} \tag{30}$$

- where $\boldsymbol{p}_{\pi c} = \frac{1}{N}\sum_{j=1}^{N}\boldsymbol{p}_{\pi j}$ is the centroid of the measured points $\{\boldsymbol{p}_{\pi j}\}_{j=1,\cdots,N}$ on the plane, and $\{\boldsymbol{p}_{\pi bk}\}_{k=1,\cdots,N_b}$ are points on the boundary (shown in Fig. 1).

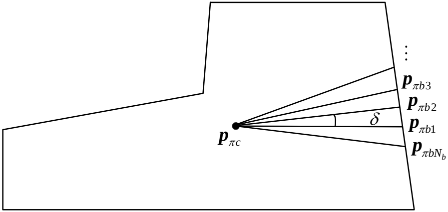


Fig. 1: Points on the boundary of the planar region.

Plane Measurement Prediction

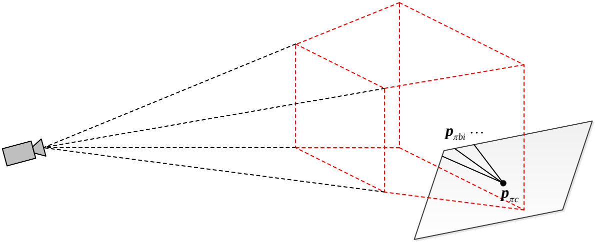
- 

Fig. 2: Predicting the measurements based on the plane map.

Constraint Analysis

- Consider a single pair of matched planes $\{{}^{c}\pi_{\pi i}, {}^{r}\pi_{\pi i}\}$. The Jacobian of the residual $\boldsymbol{e}_{\pi i}$ with respect to ξ can be computed by

$$\boldsymbol{J}_{\pi i} = \frac{\partial \boldsymbol{e}_{\pi i}}{\partial \xi} = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & (\boldsymbol{R}\cdot {}^{r}\boldsymbol{n}_i)^{\wedge} \\ (\boldsymbol{R}\cdot {}^{r}\boldsymbol{n}_i)^{T} & -\boldsymbol{t}^{T}(\boldsymbol{R}\cdot {}^{r}\boldsymbol{n}_i)^{\wedge} \end{bmatrix}. \tag{31}$$

- The camera motions that cannot be constrained by the plane pair lie in the null space of $\boldsymbol{J}_{\pi i}$, which is denoted by $\mathrm{null}(\boldsymbol{J}_{\pi i})$.

$$\mathrm{null}(\boldsymbol{J}_{\pi i}) = \left\{\xi \in \mathbb{R}^6 \mid \boldsymbol{J}_{\pi i}\xi = \boldsymbol{0}\right\} = \begin{bmatrix} \mu_1\boldsymbol{t}_1 + \mu_2\boldsymbol{t}_2 \\ \mu_3\boldsymbol{R}\cdot {}^{r}\boldsymbol{n}_i \end{bmatrix} \tag{32}$$

Constraint Analysis

- Consider the case of multiple plane pairs $\{{}^c\pi_{\pi i}, {}^r\pi_{\pi i}\}_{i=1,\cdots,N_\pi}$.

$$\Psi_\pi = \sum_{i=1}^{N_\pi} \boldsymbol{J}_{\pi i}^T \Omega_{\pi i} \boldsymbol{J}_{\pi i} \tag{33}$$

$$\Psi_\pi = \boldsymbol{Q}_\pi \Lambda_\pi \boldsymbol{Q}_\pi^T = \sum_{l=1}^{6} \lambda_{\pi l} \boldsymbol{q}_{\pi l} \boldsymbol{q}_{\pi l}^T \tag{34}$$

- $\boldsymbol{q}_{\pi l}$ forms a basis of the 6D space of a rigid motion, and $\lambda_{\pi l}$ indicates a measure of the constraint strength provided by the matched planes to the camera motion along $\boldsymbol{q}_{\pi l}$.

NBV selection

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